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Theorem: Prove that the linear span $L(S)$ of any subset S of a vector space $V(F)$ is a subspace of $V(F)$.

Proof: Taking $S = \{\alpha_1,\ \alpha_2,\ \dots,\ \alpha_m\}$ be a subset of the vector space $V(F)$.

Clearly, $L(S)$ is a nonempty subset of $V(F)$.

Let $\alpha,\ \beta\ \in L(S)$, then

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m$ and

$\beta = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_m\alpha_m$, where $a_i, b_i \in F,\ \forall\ 1 \le i \le m$.

For $a, b \in F$,

$$a\alpha + b\beta = a(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m) + b(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_m\alpha_m)$$

$$= (aa_1 + bb_1)\alpha_1 + (aa_2 + bb_2)\alpha_2 + \cdots + (aa_m + bb_m)\alpha_m$$

$$\in L(S)$$

Theorem: If S is a subset of a vector space $V(F)$, then prove that

(i). S is a subspace of $V \Leftrightarrow L(S) = S$ and (ii). $L\big(L(S)\big) = L(S)$.

Proof: Let $S = \{\alpha_1,\ \alpha_2,\ \dots,\ \alpha_m\}$ be a subset of a vector space $V(F)$.

(i) **Case 1:** Let S is a subspace of V.

Clearly $S \subseteq L(S)$.

For $\alpha \in L(S)$, $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m$, where $a_1, a_2, \dots, a_m \in F$.

Since S is a subspace of $V(F)$, then

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m \in S$, by vector addition and scalar multiplication.

This shows that $L(S) \subseteq S$ and therefore $L(S) = S$.

Case 2: Let $L(S) = S$.

We know that $L(S)$ is a subspace of a vector space $V(F)$ (By previous theorem).

This implies that S is also a subspace of a vector space $V(F)$.

(ii) We know that $L(S)$ is a subspace of a vector space $V(F)$ (Previous theorem).

Then, from (i) $L\big(L(S)\big) = L(S)$

Theorem: If S and T are the subsets of a vector space $V(F)$ then prove that

(i) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$ and (ii) $L(S \cup T) = L(S) + L(T)$.

Proof:

(i) Let $\alpha \in L(S)$. Then

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \cdots + a_n\alpha_n,$$

where $\alpha_i's \in S$ and $a_i's \in F, \forall\, 1 \le i \le n$.

Since $S \subseteq T$, so that $\alpha_i's \in T$, for all i. Then

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \cdots + a_n\alpha_n,$$

where $\alpha_i's \in T$ and $a_i's \in F, \forall\, 1 \le i \le n$.

$$\Rightarrow \alpha \in L(T)$$

i.e., $\alpha \in L(S) \Rightarrow \alpha \in L(T)$

This represents that $L(S) \subseteq L(T)$.

(ii) Let $\alpha \in L(S \cup T)$, then α can be written as

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \cdots + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + b_3\beta_3 + \cdots + b_n\beta_n,$$

where $a_i's, b_i's \in F$ and $\alpha_i's \in S, \beta_j's \in T$, for all $1 \leq i \leq m, 1 \leq j \leq n$.

Since $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \cdots + a_m\alpha_m \in L(S)$ and $b_1\beta_1 + b_2\beta_2 + b_3\beta_3 + \cdots + b_n\beta_n \in L(T)$.

Therefore $\alpha \in L(S) + L(T)$.

i.e., $\alpha \in L(S \cup T) \Rightarrow \alpha \in L(S) + L(T)$ and therefore $L(S \cup T) \subset L(S) + L(T)$ ----- (1)

Conversely, suppose that $\alpha \in L(S) + L(T)$. Then

$\alpha = \beta + \gamma$, where $\beta \in L(S)$ and $\gamma \in L(T)$.

Let $\beta = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + \cdots + c_m\alpha_m, \gamma = d_1\beta_1 + d_2\beta_2 + d_3\beta_3 + \cdots + d_n\beta_n$,

where $c_i's, d_i's \in F$ and $\alpha_i's \in S, \beta_j's \in T$, for all $1 \leq i \leq m, 1 \leq j \leq n$.

This implies $\alpha_i's, \beta_j's \in S \cup T$ and then $\alpha = \beta + \gamma \in L(S \cup T)$.

i.e., $\alpha \in L(S) + L(T) \Rightarrow \alpha \in L(S \cup T)$ and therefore $L(S) + L(T) \subset L(S \cup T)$ ----- (2)

From (1) & (2), we conclude that $L(S \cup T) = L(S) + L(T)$.

References:

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THANK YOU